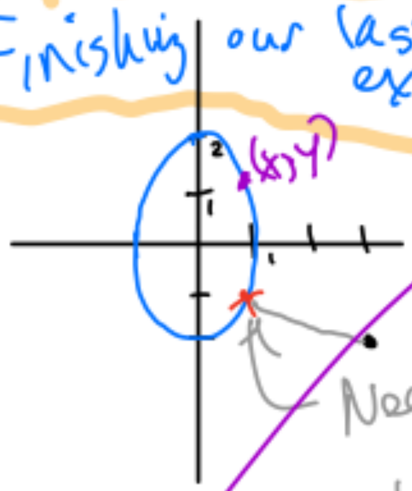
Finishing our last example



Need to find (x,y) on the graph whose slope from (x,y) to (3,-2) is the negative reciprocal of the slope of the tangent line.

4.47 Find the point on the graph of $4x^2 + y^2 = 4$ that is closest to $(3, -2)$.

slope of tangent : $8x + 2y \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-8x}{2y} = \frac{-4x}{y}$$

slope of connecting line

$$= \frac{y+2}{x-3} = \frac{y}{4x}$$

Negative reciprocal

$$4x^2 + y^2 = 4$$

solve for x & y.

$y^2 = 4 - 4x^2 = 4(1-x^2)$

$y = -2\sqrt{1-x^2}$

$$4xy + 8x = xy - 3y$$

$$\Rightarrow 3xy + 8x + 3y = 0$$

$$3x\left(-2\sqrt{1-x^2}\right) + 8x + 3\left(-2\sqrt{1-x^2}\right) = 0$$

$$-6x\sqrt{1-x^2} + 8x - 6\sqrt{1-x^2} = 0$$

$$-1 \leq x \leq 1$$

want $0 \leq x \leq 1$

$\Rightarrow x = 0.8042$ is only solution

$$y = -2\sqrt{1-x^2}$$

$$= -1.189$$

$\Rightarrow$ closest pt is $(0.8042, -1.189)$.

Back to summation formulas

Example

$$\sum_{k=1}^{6}(3-k) = (3-1)+(3-2)+(3-3)+(3-4)$$
$$+(3-5)+(3-6)$$
$$= 2+1+0+-1+-2+-3$$
$$= \boxed{-3}.$$

Useful summation formulas:

① $\displaystyle\sum_{k=1}^{n} 1 = n$

② $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\cdots+n$

Gauss trick:

$$S = 1+2+3+\cdots+100$$
$$S = 100+99+98+\cdots+1$$

add

$$2S = \underbrace{101+101+101+\cdots+101}_{100} =$$

$$2S = 100\cdot 101 \Rightarrow S = \frac{100\cdot 101}{2} = 5050$$

Similarly: $S = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

$S = n + (n-1) + (n-2) + \cdots + 1$

add $2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ terms}}$

$\Rightarrow S = \frac{n(n+1)}{2}$.

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

3. $\sum_{k=1}^{n} c f(k) = c \sum_{k=1}^{n} f(k)$ (Distributive property)

Example Find $\sum_{k=1}^{n} (7k - 2) = \sum_{k=1}^{n} 7k - \sum_{k=1}^{n} 2$

Solution: $\text{Sum} = 7\left(\sum_{k=1}^{n} k\right) - 2 \sum_{k=1}^{n} 1$

$= 7 \frac{n(n+1)}{2} - 2n$

$= \frac{7}{2}n^2 + \frac{7}{2}n - 2n = \boxed{\frac{7}{2}n^2 + \frac{3}{2}n}$

④ $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

⑤ $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Check: $\sum_{k=1}^{3} k^2 = 1 + 4 + 9 = 14$

$$\frac{3(3+1)(2(3)+1)}{6} = \frac{\cancel{3}\cdot\cancel{4}^{2}\cdot 7}{\cancel{6}} = 14. \checkmark$$

$$\sum_{k=1}^{3} k^3 = 1 + 8 + 27 = 36$$

$$\frac{n^2(n+1)^2}{4} = \frac{3^2(4)^2}{4} = 9\cdot 4 = 36. \checkmark$$

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⑥ Geometric series

$$S = a + ar + ar^2 + \cdots + ar^n$$

$$= \sum_{k=0}^{n} a\cdot r^k$$

$a = 1^{st}$ term, $n+1$ terms total

each next term is multiplied by the ratio r.

$$\Rightarrow \boxed{S = \frac{a(1-r^{n+1})}{1-r}}$$

Derivation of the formula:

$$S = a + ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$$\Rightarrow \quad rS = ar + ar^2 + ar^3 + \dots + ar^n + ar^{n+1}$$

subtract

$$S - rS = a - ar^{n+1}$$

$$(1-r)S = a(1-r^{n+1})$$

$$\Rightarrow \boxed{S = \frac{a(1-r^{n+1})}{1-r}}$$

Computing definite integrals

Find $\displaystyle\int_{-1}^{2} (2x^2-3)\,dx$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$.

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}.$$

$$x_k = a + k\Delta x = -1 + \frac{3k}{n}$$

$$f(x_k) = 2x_k^2 - 3 = 2\cdot\left(-1+\frac{3k}{n}\right)^2 - 3$$

$$\int_{-1}^{2}(2x^2-3)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(2\cdot\left(-1+\frac{3k}{n}\right)^2-3\right)\cdot\frac{3}{n}$$

$$=\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\left(2\left(1-\frac{6k}{n}+\frac{9k^2}{n^2}\right)-3\right)$$

$$2-\frac{12k}{n}+\frac{18k^2}{n^2}-3$$

$$=\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}-1-\frac{12k}{n}+\frac{18k^2}{n^2}$$

$$=\lim_{n\to\infty}\frac{3}{n}\left(-1\cdot\underbrace{\left(\sum_{k=1}^{n}1\right)}_{n}-\frac{12}{n}\underbrace{\left(\sum_{k=1}^{n}k\right)}_{\frac{n(n+1)}{2}}+\frac{18}{n^2}\underbrace{\left(\sum_{k=1}^{n}k^2\right)}_{\frac{n(n+1)(2n+1)}{6}}\right)$$